Weeks of September 8th and 15th

## Algebraic structures, ℕ and ℤ, prime numbers
## Crystals and Symmetries

## Additional Exercises

**Exercise 1: Sub-groups** (more abstract, just for fun)

Let's consider a group G for an operation $*$, and two subgroups A and B of G. Show that $A \cup B$ is a subgroup of G if and only if $A \subseteq B$ *or* $B \subseteq A$.

Hint on notations: $A \cup B$, or A union B, is a set that gathers all the elements of both groups. $A \subseteq B$, or A included in B means that all the elements of A also belong to B.

**Exercise 2: The Fibonacci sequence**

The Fibonacci sequence first appears in his Liber Abaci (The Book of Calculation, 1202) where he calculates the growth of rabbit populations. Assume that:

- A newly born breeding pair of rabbits are put in a field at month 1;
- Each breeding pair mates at the age of one month, and at the end of their second month they always produce another pair of rabbits;
- Rabbits never die, but continue breeding forever.

At month $n \in \mathbb{N}$, we call $F_n$ the number of pairs of rabbits ($F_0 = 0$ & $F_1 = 1$).

2a. How many pairs of rabbit will there be at months two, three and four?

2b. For $n \geq 2$, explain why: $F_n - F_{n-1} = F_{n-2}$

2c. For $n \in \mathbb{N}$, find the two real solutions $\varphi > 0$ and $\psi < 0$ of the equation:

$$x^{n+2} = x^{n+1} + x^n.$$

2d. We define the sequence $u_n = a\varphi^n + b\psi^n$, with $(a, b) \in \mathbb{R}^2$. Show that:

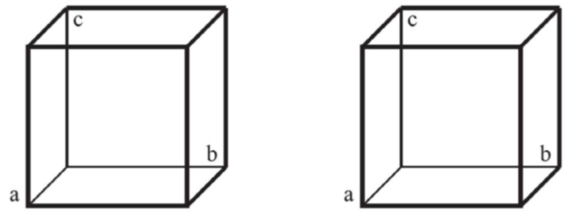
$$\forall n \in \mathbb{N}, u_{n+2} = u_{n+1} + u_n$$

2e. Express $(a, b) \in \mathbb{R}^2$ as a function of $\varphi$ and $\psi$ such that $u_0 = 0$ and $u_1 = 1$

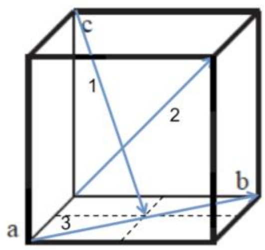
2f. Conclude that $\forall n \in \mathbb{N}, F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

**Exercise 3: Miller indices**

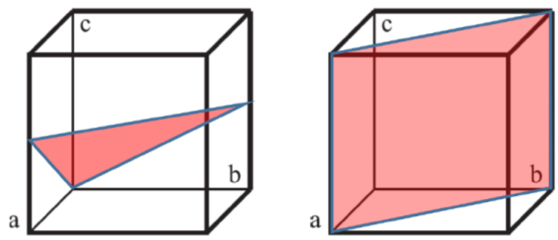
3a. Draw the crystal directions [101], [$1\bar{1}2$] in the cubes to the right representing the cubic structure:




3b. Find the Miller indices of the directions (1, 2 and 3) shown in the cube on the right.
What is the angle between directions 1 and 3 ?
To which plan do directions 1 and 3 both belong ?




2c. What are the Miller indices of the crystal planes below:



**Exercise 4: Diamond structure**

The diamond structure shown below consists of tetrahedra of carbon atoms arranged in space to form the crystal. This arrangement turns out to be represented by a motif of two carbon atoms translated in the face-centered cubic Bravais lattice. The atoms can be represented by a sphere of radius $R$. In the motif, one atom has its center at one corner of the cube that could be an origin O (0,0,0), and the other one is shifted along the diagonal at position (¼, ¼, ¼). The two spheres representing the two atoms are in contact with each other (to the contrary of what is shown on the schematic below for clarity).

4a. Is the point of coordinate (¼, ¼, ¼) a lattice point ?

4b.

(i) How many atoms are in the conventional FCC cell for Diamond ?

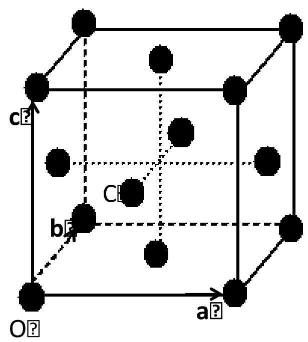
(ii) What is the coordination number ?

4c. Deduce the packing fraction of the Diamond structure.

4d. Find a condition on mutually prime relative integers (h,k,l) so that the plan (hkl) goes through the point P(¼, ¼, ¼).

**Exercise 5 : Primitive cell of the FCC structure**

Consider the Face-centered cubic conventional cell shown to the right with the origin marked as O and the three orthogonal axis of length the cube edge $a$.




5a. In the basis (O, **a**, **b**, **c**), what are the coordinate of the point C at the center of a face of the cube ? Is this basis a Bravais lattice for the FCC cubic structure ?

5b. We consider the same origin O and three new vectors **a'**, **b'** and **c'** such that :

$$\vec{a}' = \frac{1}{2}(\vec{b} + \vec{c})\ ;\ \vec{b}' = \frac{1}{2}(\vec{a} + \vec{c})\ ;\ \vec{c}' = \frac{1}{2}(\vec{a} + \vec{b})$$

(i) Show that for all point M of the FCC lattice, one can find relative integers $n$, $p$ and $q$, such that: $\overrightarrow{OM} = \frac{n}{2}\vec{a} + \frac{p}{2}\vec{b} + \frac{q}{2}\vec{c}$, with *(n,p,q)* either all even numbers, or 1 is odd and 2 are even numbers.

(ii) Express the **OM** vector in the (O, **a’**, **b’**, **c’**) basis and conclude whether this basis is indeed a primitive basis for the FCC structure.

5c. Calculate the volume occupy by the FCC primitive cell in two ways:

(i) Using the vectorial formula seen in class;

(ii) Making an argument regarding the number of motifs should be present in the primitive and conventional cells of the FCC.